CuO_4 and $YBa_2Cu_3O_7$," *Physical Review B,* Vol. 36, No. 10, 1987, pp. 5634-5685.

[5]Heremans, J., Morelli, D. T., Smith, G. W., and Strite, III, S. C., "Thermal and Electronic Properties of Rare-Earth Ba_2Cu_3Ox Superconductors," *Physical Review B,* Vol. 37, No. 4, 1988, pp. 1604-1610.

[6]Tajima, S., Uchida, S., Tanaka, S., Kanbe, S., Kitazawa, K., and Fueki, K., "Plasma Spectra of New Superconductors $(La_{1-x}A_x)_2$-CuO_4," *Japanese Journal Applied Physics,* Vol. 26, 1987, pp. L432-L433.

[7]Pandey, R. K., Gilbert, G. R., Kirk, W. R., Kobiela, P. S., Clearfield, A., and Squattrito, P. J., "Processing of Single-Phase Ceramic 1-2-3 YBaCu-Oxide Superconductor by Hot Pressing," *Journal of Superconductivity,* Vol. 1, No. 1, 1988, pp. 45-52.

[8]Cunnington, G. R., Bell, G. A., Armaly, B. F., and Tien, C. L., "Total Emittance Measurements of Thin Metallic Films at Cryogenic Temperatures," *Journal of Spacecraft and Rockets,* Vol. 7, No. 12, 1970, pp. 1496-1499.

View Factor Between Differing-Diameter, Coaxial Disks Blocked by a Coaxial Cylinder

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Introduction

VIEW-factor algebra can be used to derive the view factors for complicated geometries from available view factors for simpler geometries.[1-3] This approach is particularly useful for the numerical analysis of heat-transfer problems involving diffuse-gray radiative interchange among the surfaces of the system, where finite-element and finite-difference methods require calculation of the view factors between the many surfaces which correspond to the numerical grid.

Radiative heat transfer is extremely important in Czochralski crystal growth of semiconductor materials.[4] Here a cylindrical crystal is grown from its melt by withdrawing the crystal and adjusting the heat transfer needed for solidification. When the radiative transfer is diffuse and gray and the crystal is a perfect cylinder, the necessary view factors can be derived using view-factor algebra in conjunction with view factors for three simple geometries available in the literature.[5] The computational effort is reduced if these view factors are available as algebraic expressions and are not evaluated numerically.

One particular view factor needed for the Czochralski configuration is that between parallel, coaxial disks of different diameters that are separated by a solid, coaxial cylinder (see Fig. 1). Minning[6] and Holchendler and Laverty[7] both attempted to derive this view factor. (Holchendler and Laverty's analysis is for a frustrum, which is a cylinder in the limit when the diameters at its two ends are equal, separating the two opposing disks.) They both succeeded in deriving the view factor from a differential surface to an opposing finite disk. However, neither was able to integrate this expression to obtain a closed-form result for the view factor from a finite disk to an opposing finite disk because of the complexity of the resulting integrand. In this Note, a closed-form expression for this view factor is presented that is valid when the disks both have finite radius.

Analysis

The expression presented by both Minning[6] and Holchend- ler and Laverty[7] for the view factor from a differential surface to an opposing finite disk is derived using the contour integral method[8]:

$$F_{dA_\rho-A_a}=\frac{\cos^{-1}\frac{c}{a}}{2\pi}-\frac{c^2-\rho^2-h^2}{\pi\sqrt{(c^2+\rho^2+h^2)^2-4\rho^2c^2}}\tan^{-1}\left[\sqrt{\frac{(c^2+\rho^2+h^2+2\rho c)(\rho-c)}{(c^2+\rho^2+h^2-2\rho c)(\rho+c)}}\right]+\frac{a^2-\rho^2-h^2}{\pi\sqrt{(a^2+\rho^2+h^2)^2-4\rho^2a^2}}$$
$$\times\tan^{-1}\left[\sqrt{\frac{(a^2+\rho^2+h^2+2a\rho)(\rho a-c^2+\sqrt{\rho^2-c^2}\sqrt{a^2-c^2})}{(a^2+\rho^2+h^2-2a\rho)(\rho a+c^2-\sqrt{\rho^2-c^2}\sqrt{a^2-c^2})}}\right] \tag{1}$$

Here a is the radius of the finite disk, c is the radius of the solid cylinder, h is the height of the cylinder, which is also the axial distance between the finite disk and the plane defined by the differential surface, and ρ is the distance between the differential surface and the vertical axis. The view factor $F_{dA_\rho-A_a}$ is from the differential surface with area dA_ρ to the finite disk with outer radius a, inner radius c, and area A_a. Equation (1) is equivalent to Eq. (6) in Minning's paper and to Eq. (13) in Holchendler and Laverty's paper. There are some sign mistakes in Eq. (6) in Minning's paper.

The view factor from a finite disk with outer radius b and inner radius c to a disk with outer radius a and inner radius c is obtained by integrating Eq. (1) as

$$A_bF_{A_b-A_a}=2\pi\int_c^b F_{dA_\rho-A_a}\rho\,d\rho \tag{2}$$

where $F_{A_b-A_a}$ is the view factor from a disk with outer radius b, inner radius c, and area A_b to a coaxial, parallel disk with outer radius a, inner radius c, and area A_a. The disks are separated by a solid, coaxial cylinder of radius c and height h. Substituting

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Eq. (1) into Eq. (2) and integrating by parts gives

$$A_bF_{A_b-A_a} = \int_c^b \rho \cos^{-1}\frac{c}{a}\mathrm{d}\rho + \sqrt{(c^2+\rho^2+h^2)^2-4\rho^2c^2} \times \tan^{-1}\left[\sqrt{\frac{(c^2+\rho^2+h^2+2\rho c)(\rho-c)}{(c^2+\rho^2+h^2-2\rho c)(\rho+c)}}\right]\Bigg|_c^b$$

$$-\int_c^b \frac{c\{[(c^2+\rho^2+h^2)^2-4\rho^2c^2]+2(\rho^2-c^2)(c^2-\rho^2+h^2)\}}{2\rho\sqrt{\rho^2-c^2}(\rho^2-c^2+h^2)}\mathrm{d}\rho - \sqrt{(a^2+\rho^2+h^2)^2-4\rho^2a^2}$$

$$\times \tan^{-1}\left[\sqrt{\frac{(a^2+\rho^2+h^2+2\rho a)(\rho a-c^2+\sqrt{\rho^2-c^2}\sqrt{a^2-c^2})}{(a^2+\rho^2+h^2-2\rho a)(\rho a+c^2-\sqrt{\rho^2-c^2}\sqrt{a^2-c^2})}}\right]\Bigg|_c^b$$

$$+\int_c^b \frac{c\{[(a^2+\rho^2+h^2)^2-4\rho^2a^2](\sqrt{a^2-c^2}+\sqrt{\rho^2-c^2})+2\sqrt{\rho^2-c^2}(\sqrt{a^2-c^2}+\sqrt{\rho^2-c^2})^2(a^2-\rho^2+h^2)\}}{2\rho\sqrt{\rho^2-c^2}(\sqrt{a^2-c^2}+\sqrt{\rho^2-c^2})[(\sqrt{a^2-c^2}+\sqrt{\rho^2-c^2})^2+h^2]}\mathrm{d}\rho \tag{3}$$

The latter two integrals in Eq. (3) are simplified to

$$-\int_c^b \frac{c[(h^2+c^2-\rho^2)(h^2-c^2+\rho^2)+4\rho^2h^2]}{2\rho\sqrt{\rho^2-c^2}(\rho^2-c^2+h^2)}\mathrm{d}\rho$$

$$+\int_c^b \frac{c[4\rho^2h^2+(a^2-\rho^2+h^2)(\rho^2+a^2+h^2-2c^2+2\sqrt{a^2-c^2}\sqrt{\rho^2-c^2})]}{2\rho\sqrt{\rho^2-c^2}[(\sqrt{a^2-c^2}+\sqrt{\rho^2-c^2})^2+h^2]}\mathrm{d}\rho$$

When common factors are canceled, these integrals are evaluated to yield

$$\frac{a^2-c^2}{2}\cos^{-1}\left(\frac{c}{\rho}\right)\Bigg|_c^b + 2ch\left[\tan^{-1}\left(\frac{\sqrt{\rho^2-c^2}+\sqrt{a^2-c^2}}{h}\right)-\tan^{-1}\left(\frac{\sqrt{\rho^2-c^2}}{h}\right)\right]\Bigg|_c^b$$

Substituting these terms into Eq. (3) and evaluating its limits gives

$$A_bF_{A_b-A_a} = \frac{a^2-c^2}{2}\cos^{-1}\frac{c}{b}+\frac{b^2-c^2}{2}\cos^{-1}\frac{c}{a}$$

$$-\sqrt{(a^2+b^2+h^2)^2-4a^2b^2}\tan^{-1}\left[\sqrt{\frac{(a^2+b^2+h^2+2ab)(ba-c^2+\sqrt{b^2-c^2}\sqrt{a^2-c^2})}{(a^2+b^2+h^2-2ab)(ba+c^2-\sqrt{b^2-c^2}\sqrt{a^2-c^2})}}\right]$$

$$+\frac{1}{2}\sqrt{(a^2+c^2+h^2)^2-4a^2c^2}\cos^{-1}\left[\frac{c(h^2-a^2+c^2)}{a(h^2+a^2-c^2)}\right]+\frac{1}{2}\sqrt{(b^2+c^2+h^2)^2-4b^2c^2}\cos^{-1}\left[\frac{c(h^2-b^2+c^2)}{b(h^2+b^2-c^2)}\right]$$

$$+2ch\left[\tan^{-1}\left(\frac{\sqrt{b^2-c^2}+\sqrt{a^2-c^2}}{h}\right)-\tan^{-1}\left(\frac{\sqrt{b^2-c^2}}{h}\right)-\tan^{-1}\left(\frac{\sqrt{a^2-c^2}}{h}\right)\right] \tag{4}$$

where

$$A_b = \pi(b^2-c^2)$$

The trigonometric identity

$$2\tan^{-1}\sqrt{u} = \cos^{-1}\frac{1-u}{1+u}$$

is applied to simplify the fourth term on the right-hand side of Eq. (4).

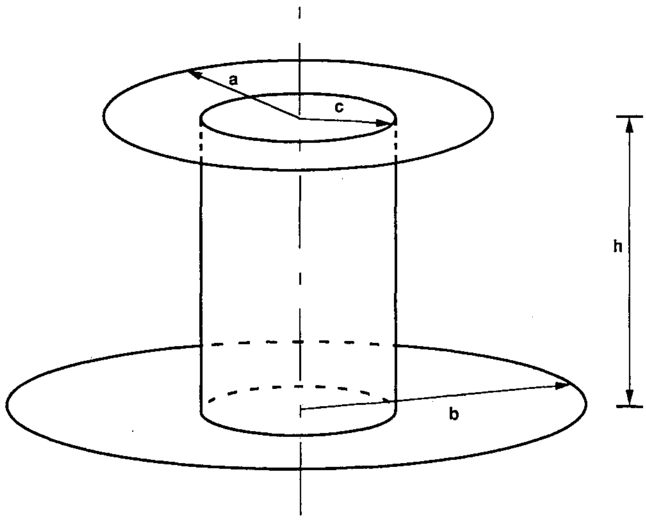

Fig. 1 Schematic of geometry for view factor from disk to coaxial, parallel disk with coaxial blocking cylinder.

The correctness of Eq. (4) was checked by numerical integration of Eqs. (1) and (2), which agrees with the closed-form result. Moreover, Eq. (4) simplifies to various limiting cases for which analytical solutions are available. When the radius of the blocking cylinder c approaches zero, Eq. (4) yields the expression for the view factor from one disk to another coaxial, parallel disk.[2] Leuenberger and Person[9] derived an expression for the view factor between parallel, coaxial disks with equal outer radii b and inner radii c separated by a solid coaxial cylinder with radius c and height h, and Eq. (4) is simplified to this result when the disk radius a is set equal to the disk radius b. Finally, when the radius a in Eq. (4) is allowed to increase without bound, the view factor from the disk with radius b to the opposing disk becomes

$$F_{A_b - A_{a-\infty}} = 1 - F_{A_b - A_{\text{cyl}}}$$

where $F_{A_b - A_{\text{cyl}}}$ is the view factor[10] from the disk with outer radius b, inner radius c, and area A_b to the coaxial cylinder with radius c, height h, and area A_{cyl}. As the area of the opposing disk becomes unbounded, all radiation leaving the disk with radius b must either strike the coaxial cylinder or the opposing disk.

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References

[1]Hamilton, D. C., and Morgan, W. R., "Radiant-Interchange Configuration Factors," NACA TN-2836, 1952.

[2]Sparrow, E. M., and Cess, R. D., *Radiation Heat Transfer*, augmented ed., McGraw-Hill, New York, 1978, pp. 136–141; 340–342.

[3]Siegel, R., and Howell, J. R., *Thermal Radiation Heat Transfer*, 2nd ed., McGraw-Hill, New York, 1981, pp. 190–202.

[4]Brown, R. A., "Theory of Transport Processes in Single Crystal Growth from the Melt," *AIChE Journal*, Vol. 34, June 1988, pp. 881–911.

[5]Srivastava, R. K., Ramachandran, P. A., and Dudukovic, M. P., "Radiation View Factors in Czochralski Crystal Growth Apparatus for Short Crystals," *Journal of Crystal Growth*, Vol. 74, Feb.–March 1986, pp. 281–291.

[6]Minning, C. P., "Shape Factors Between Coaxial Annular Disks Separated by a Solid Cylinder," *AIAA Journal*, Vol. 17, March 1979, pp. 318–320.

[7]Holchendler, J., and Laverty, W. F., "Configuration Factors for Radiant Heat Exchange in Cavities Bounded at the Ends by Parallel Disks Having Conical Centerbodies," *ASME Journal of Heat Transfer*, Vol. 96, May 1974, pp. 254–257.

[8]Sparrow, E. M., "A New and Simpler Formulation for Radiative Angle Factors," *ASME Journal of Heat Transfer*, Vol. 85, May 1963, pp. 81–88.

[9]Leuenberger, H., and Person, R. A., "Compilation of Radiation Shape Factors for Cylindrical Assemblies," American Society of Mechanical Engineers, Paper 56-A-144, 1956.

[10]Rea, S. N., "Rapid Method for Determining Concentric Cylinder Radiation View Factors," *AIAA Journal*, Vol. 13, Aug. 1975, pp. 1122–1123.

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